2019-Sep-24 (lecture) MAT B44-2019 Prof. Yun William Yu

Tuesday, September 24, 2019 3:04 PM

Outline:

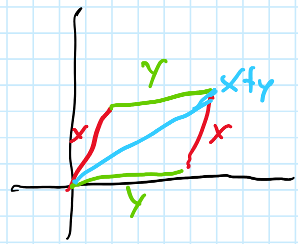
0. Norms and fixed points
1. Proof: Contraction → fixed pt
2. Examples of contractions
3. Picard iteration

Last time:

We gave examples of norms, fixed pts, and contractions.

A **norm** on a vector space X has to satisfy 3 properties

1) $\|0\| = 0$. $\|x\| > 0$ for $x \neq 0$.
2) $\|\alpha x\| = |\alpha| \|x\|$ for $\alpha \in \mathbb{R}$ and $x \in X$
3) $\|x+y\| \leq \|x\| + \|y\|$ for $x, y \in X$



This time:

Claim: Also implies the inverse triangle inequality $\big| \|x\| - \|y\| \big| \leq \|x-y\|$

proof. $\|x\| = \|(x-y)+y\| \leq \|x-y\| + \|y\|$

$\|x\| - \|y\| \leq \|x-y\|$

Similarly $\|y\| - \|x\| \leq \|y-x\| = \|x-y\|$

$\Rightarrow \big| \|x\| - \|y\| \big| \leq \|x-y\|$ $\square$

Examples of norms

Ex. $X = \mathbb{R}$, $\|x\| = |x|$

1) $|0| = 0$, $|x| > 0$, if $x \neq 0$.
2) $|\alpha x| = |\alpha| |x|$
3) $|x+y| \leq |x| + |y|$

$\Bigg\}$ Showing these 3 properties implies absolute value is a norm on $\mathbb{R}$.

Ex. $X = C(I)$. I is a compact interval on $\mathbb{R}$
(e.g. $I = [0,1]$)

$$\|x\| = \sup_{t \in I} |x(t)|$$

Note that the **supremum** is the least upper bound of a set.

i.e. Given a set $A \subseteq \mathbb{R}$, define $B = \{ b \mid b \geq a \ \ \forall a \in A \}$

Then $\sup(A) = \min(B)$.

e.g. $A = (0,1)$, $\sup(A) = 1$

$A = [0,1]$, $\sup(A) = 1$

$A = \{0, 1, 2, 3\}$ $\sup(A) = 3$

$A = \mathbb{R}$, $\sup(A)$ is undefined (sometimes, we use the extended real line, $\sup(A) = +\infty$)

A few properties:

- $\sup(A+B) \equiv \sup(\{a+b \mid a \in A, \ b \in B\}) = \sup(A) + \sup(B)$
- If $x: I \to \mathbb{R}$, $y: I \to \mathbb{R}$,

$$\sup_{t \in I}(x(t) + y(t)) \leq \sup_{t \in I} x(t) + \sup_{t \in I} y(t)$$

- $\sup(\alpha A) \equiv \sup(\{\alpha a \mid a \in A\}) = \alpha \sup(A)$

Ex.
- $x(t) = t^2 + 1$, and $I = [0,1]$

$$\|x\| = \sup_{t \in [0,1]} |t^2+1| = 2$$

- $x(t) = -t - 5$, and $I = [0,1]$

$$\|x\| = \sup_{t \in [0,1]} |-t-5| = 6.$$

- $x(t) = \sin(t) - 1$ and $I = [0, 1000000]$

$$\|x\| = \sup_{t\in I} |\sin(t) - 1| = 2$$

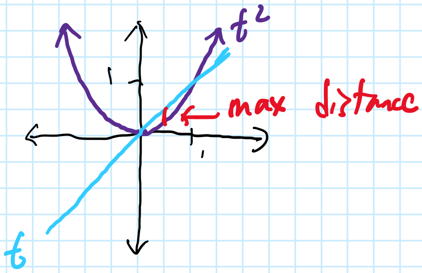
continuous functions on I

Note The function $\|x\| = \sup_{t\in I} |x(t)|$ is a norm on $C(I)$ (HW)

This means that we can define "distances" b/t functions

The "distance" between two functions is the maximum distance between them on a closed interval.

Ex. $x(t) = t^2$, $y(t) = t$, $I = [0,1]$



$$\|x-y\| = \sup_{t\in I} |t^2 - t|$$

Max distance is at a critical pt of $t^2 - t$

Let $g(t) = t^2 - t$

$g'(t) = 2t - 1$

$\Rightarrow t = \frac{1}{2}$ at crit pt.

$g(\frac{1}{2}) = -\frac{1}{4}$

$$\Rightarrow \sup_{t\in I} |t^2 - t| = \frac{1}{4}$$

A sequence of functions $x_n(t)$ converges to $x(t)$ if and only if (iff)

$$\lim_{n\to\infty} \|x_n - x\| = \lim_{n\to\infty} \sup_{t\in I} |x_n(t) - x(t)| = 0$$ (uniform convergence)

Turns out (real analysis) that a Cauchy sequence under this metric converges to another continuous function, so the space is complete.

Thus, the vector space $C(I)$, together with $\|x\| = \sup_{t\in I} |x(t)|$ (for I a compact interval) is a Banach space.

We can now reason about $C(I)$ using tools of Banach spaces

Def. A fixed point of a mapping $K: C \subseteq X \to C$ is an element $x \in C$ s.t. $K(x) = x$.

Def. A **contraction** is a mapping $K: C \subseteq X \to C$ where there exists a contraction constant $\theta \in [0,1)$ s.t.

$$\|K(x) - K(y)\| \leq \theta \|x - y\|, \quad x, y \in C.$$

Notation: $K^n(x) = K(K^{n-1}(x))$, and $K^0(x) = x$.

Last time we had a function

$K(x) = 1000 + \frac{x}{2}$, which we informally called a contraction

More formally, let $C = [0, 10^6]$.

Note $K(x) \in C$ for all $x \in C$, so it is a mapping $K: C \subseteq X \to C$.

The contraction constant is $\theta = \frac{1}{2}$ because

$$\|K(x) - K(y)\| = \|1000 + \tfrac{x}{2} - (1000 + \tfrac{y}{2})\| = \|\tfrac{x}{2} - \tfrac{y}{2}\| = \tfrac{1}{2}\|x - y\|.$$

So, $K(x) = 1000 + \frac{x}{2}$ is a contraction on $C = [0, 10^6]$

What about $C = [0, 10]$?

No. Because $K(0) = 1000 \notin [0, 10]$

Last time, we had a function

$K(x) = \sqrt{x}$

Is it a contraction on $C = [0, 1]$?

No. Because $|\sqrt{0} - \sqrt{1}| = |0 - 1| = 1$, so it doesn't contract them.

What about on $C = [0.5, 1.5]$?

Yes. First note that $0.5 \leq \sqrt{x} \leq 1.5$ if $x \in [0.5, 1.5]$

Also, for $x \geq 0.5$, $\frac{1}{\sqrt{x} + \sqrt{y}} \leq \frac{1}{\sqrt{0.5} + \sqrt{0.5}} = \frac{1}{2\sqrt{\frac{1}{2}}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.707$

Now $|\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{1}{\sqrt{2}}|x - y|.$

Thus, $K(x) = \sqrt{x}$ is a contraction on $C = [0.5, 1.5]$.

When checking a contraction, make sure to check that the mapping

1) Puts points in C closer together by a multiplication factor

mapping

1) Puts points in C closer together by a multiplicative factor
2) maps into the subset C.

Theorem: (Tesch 2.1) Banach fixed pt theorem (Contractive principle)

Let C be a (nonempty) closed subset of a Banach space X and let $K: C \to C$ be a contraction. Then K has a unique fixed point $\bar{x} \in C$ s.t.

$$\| K^n(x) - \bar{x} \| \leq \frac{\theta^n}{1-\theta} \| K(x) - x \| \quad , \; x \in C.$$

Proof. See pre-lecture notes, or Thursday lecture.

As long as we have a contraction in a closed subset of a Banach space, there exists a unique fixed pt.

Let's apply the contraction principle to some examples

Ex. Consider a mapping $K: \mathbb{R} \to \mathbb{R}$, defined by $K(x) = 1000 + \frac{x}{2}$. Prove that for any $x \in \mathbb{R}$, $\lim_{n\to\infty} K^n(x) = 2000$.

In order to use the contraction principle, we need a closed subset $C \subseteq \mathbb{R}$ s.t. K is a contraction on $\mathbb{R}$.

$\mathbb{R}$ is a closed subset of $\mathbb{R}$, and $|K(x) - K(y)| = \frac{1}{2}|x-y|$ for all $x, y \in \mathbb{R}$.

So there exists exactly one fixed pt.

Note, $K(2000) = 2000$, so $\bar{x} = 2000$ is a fixed pt.

Thus, all starting points in $\mathbb{R}$ converge to 2000 under repeated iterations of K.

Ex. Consider a mapping $K: \mathbb{R} \to \mathbb{R}$, $K(x) = \sqrt{x}$.

Prove that for any $x_0 \geq 0.5$, $\lim_{n\to\infty} K^n(x_0) = 1$.

In order to use the contraction principle, we need a closed subset of $\mathbb{R}$ on which K is a contraction.

Obviously, $\mathbb{R}$ doesn't work,

Lemma: If $f: C \subseteq \mathbb{R} \to C$ has a continuous derivative on a closed interval C, and $\forall x, |f'(x)| \le \theta$ for some constant $\theta < 1$, then f is a contraction.

proof. Let $x, y \in C$, By the Mean Value theorem, $\exists\, c \in [x, y]$ s.t. $|f(x) - f(y)| = |f'(c)||x-y|$.

But $f'(c) \le \theta < 1$, so $|f(x) - f(y)| \le \theta |x-y|$, $\square$

Back to $K(x) = \sqrt{x}$. Note that $K'(x) = -\frac{1}{2\sqrt{x}}$.

Thus, $|K'(x)| = \frac{1}{2\sqrt{x}} \le \frac{1}{2\sqrt{\frac{1}{2}}}$ for all $x \ge \frac{1}{2}$.

$\Rightarrow |K'(x)| \le \frac{\sqrt{2}}{2} \approx 0.707$ for all $x \ge \frac{1}{2}$.

Also $K(x) \in [0.5, \infty)$ for all $x \in [0.5, \infty)$. Thus $K(x)$ is a contraction on $C = [0.5, \infty)$.

Since $K(1) = 1$ is a fixed point, $\lim_{n \to \infty} K^n(x) = 1$ for all $x \ge 0.5$.

Recall: We proved earlier that $C(I)$ is a Banach space of functions.

Now we just need to find a contraction whose fixed pt is the solution to an ODE.

Picard iteration:

Consider an initial value problem (IVP)

$$\dot{x} = f(t, x), \quad x(t_0) = x_0,$$

where $x, t \in \mathbb{R}$, and $f \in C(U, \mathbb{R})$ where $U \subseteq \mathbb{R}^2$ is an open subset of $\mathbb{R}^2$ and $(t_0, x_0) \in U$.

Note: We consider here $x \in \mathbb{R}$. The proof we give also works with open subset of $\mathbb{R}^2$ and $(t_0, x_0) \in U$.

Note: We consider here $x \in \mathbb{R}$. The proof we give also works with minor modifications for $x \in \mathbb{R}^n$, $f \in C(U, \mathbb{R}^n)$, where $U \subseteq \mathbb{R}^{n+1}$. This means that existence and uniqueness will hold for all first-order systems. At the beginning of term, we showed that all higher-order systems can be transformed into equivalent first-order systems. Thus, this existence and uniqueness proof will apply to all ODEs under certain technical conditions.

Let's integrate both sides with respect to t,

$$\dot{x} = f(t,x)$$

$$\int_{s=t_0}^{s=t} \dot{x}(s)\,ds = \int_{s=t_0}^{s=t} f(s, x(s))\,ds$$

$$x(t) - x(t_0) = \int_{t_0}^{t} f(s, x(s))\,ds$$

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds.$$

This integral equation is equivalent to our original ODE i.e. if for some function $x(t)$, $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds$, then $x(t)$ is a solution to the original $\dot{x} = f(t,s)$.

Let's define **Picard iteration** by a map $K : C(U, \mathbb{R}) \to C(U, \mathbb{R})$

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds.$$

And the **Picard iterates**

$$x_0(t) = x_0 \qquad \text{(the constant function through the scalar } x_0\text{)}$$

$$x_1(t) = K(x_0)(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\,ds$$

$$x_2(t) = K^2(x_0)(t) = K(x_1)(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))\,ds$$

$$x_2(t) = K^2(x_0)(t) = K(x_1)(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))\,ds$$

$$x_3(t) = K^3(x_0)(t) = K(x_2)(t) = x_0 + \int_{t_0}^{t} f(s, x_2(s))\,ds$$

$$\vdots$$

$$x_m(t) = K^m(x_0)(t) = K(x_{m-1})(t) = x_0 + \int_{t_0}^{t} f(s, x_{m-1}(s))\,ds$$

The solution $x(t)$ is a fixed pt. under Picard iteration, so if we can prove Picard iteration is a contraction, then that would immediately imply existence and uniqueness.

But first, let's try a few examples of Picard iteration.

Ex. $\dot{x}(t) = tx$, and $x(0) = 1$.

$$\frac{dx}{dt} = tx$$

$$\frac{dx}{x} = t\,dt$$

$$\ln x = \frac{1}{2}t^2 + C$$

$$x = Ce^{\frac{1}{2}t^2}$$

$$x(0) = 1 \implies C = 1$$

solution is

$$x(t) = e^{t^2/2}.$$

The Picard iterates are (let $f(t,x) = tx$)

$$x_0(t) = x_0 = 1$$

$$x_1(t) = 1 + \int_0^t s\,x_0(s)\,ds = 1 + \int_0^t s\,ds = 1 + \frac{t^2}{2}.$$

$$x_2(t) = 1 + \int_0^t s\left(1 + \frac{s^2}{2}\right)ds = 1 + \frac{t^2}{2} + \frac{t^4}{8}.$$

$$x_3(t) = 1 + \int_0^t s\left(1 + \frac{s^2}{2} + \frac{s^4}{8}\right)ds = 1 + \frac{t^2}{2} + \frac{t^4}{8} + \frac{t^6}{48}$$

Note that $e^{a} = 1 + a + \frac{a^2}{2} + \frac{a^3}{3!} + \frac{a^4}{4!} + \cdots$

$$\text{So } e^{t^2/2} = 1 + \frac{t^2}{2} + \frac{t^4}{8} + \frac{t^6}{48} + \frac{t^8}{384} + \cdots$$

So, the Picard iterates are slowly approximating the true solution.

Ex. $\dot{x}(t) = t - x$, where $x(1) = 2$

$$x_0 = 2$$

$$x_1 = 2 + \int_1^t (s - x_0)\,ds = 2 + \int_1^t (s-2)\,ds = 2 + \left[\frac{s^2}{2} - 2s\right]_1^t$$

$$= \frac{t^2}{2} - 2t + 2 - \left(\frac{1}{2} - 2\right) = \frac{t^2}{2} - 2t + \frac{7}{2}.$$

$$x_2 = 2 + \int_1^t (s - x_1)\,ds = 2 + \int_1^t \left(s - \frac{s^2}{2} + 2s - \frac{7}{2}\right) ds$$

$$= 2 + \int_1^t \left(-\frac{s^2}{2} + 3s - \frac{7}{2}\right) ds = 2 + \left[-\frac{s^3}{6} + \frac{3}{2}s^2 - \frac{7}{2}s\right]_1^t$$

$$= 2 + \left(-\frac{t^3}{6} + \frac{3t^2}{2} - \frac{7}{2}t\right) - \left(-\frac{1}{6} + \frac{3}{2} - \frac{7}{2}\right)$$

$$= 2 + \frac{13}{6} - \frac{t^3}{6} + \frac{3t^2}{2} - \frac{7}{2}t = -\frac{t^3}{6} + \frac{3t^2}{2} - \frac{7}{2}t + \frac{25}{6}.$$